

Negative time delay for wave reflection from a one-dimensional semi-harmonic well

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To Professor Bogdan Mielnik with our deepest admiration.

Abstract It is reported that the phase time of particles which are reflected by a one-dimensional semi-harmonic well includes a time delay term which is negative for definite intervals of the incoming energy. In this interval, the absolute value of the negative time delay becomes larger as the incident energy becomes smaller. The model is a rectangular well with zero potential energy at its right and a harmonic-like interaction at its left.

The time taken by a particle to traverse a given spatial region is one of the most striking features of quantum theory [1, 2]. In the case of tunneling through a one-dimensional barrier of height V_0 and width ξ , the *transmission time* of a wave packet centered at the average total energy $E = \hbar\omega = \hbar^2 k^2 / (2m) < V_0$ is independent of the barrier thickness [3]. Thus, the peak value of the packet propagates with the effective group velocity $v_g = d\omega/dk = \hbar k/m$, which must increase with ξ across the barrier. Using electromagnetic analogues, superluminal (“anomalously large”) group velocities have been observed for evanescent modes [4], microwave pulses [5], and in the tunneling of photons through 1D photonic band gaps [6]. Indeed, this ‘abnormal behavior’ of light [7] has stimulated the designing of high-speed devices based on the tunneling properties of semiconductors (see, e.g., Chs. 11 and 12 of Ref. [1]). In the stationary phase approximation [8], the *phase time* (group delay) is defined as the energy derivative of the transmission phase $\tau_W = \hbar \frac{d\varphi}{dE} = \frac{1}{v_g} \frac{d\varphi}{dk}$. This gives information of the time taken by the peak of the transmitted packet to appear, measured from the moment the peak of the incident packet strikes a given barrier. Another well established notion of time considers the average time spent by the particles in the barrier. It is called the *dwell time* and is defined as the ratio $\tau_D = n/j$, with n the number of particles within the barrier and j the incident flux [9]. Yet, τ_W and τ_D are not necessarily related with each other; they are comparable only if the barrier is almost transparent [10].

While the quantum tunneling of rectangular barriers has attracted a lot of attention in recent years (see e.g., [11, 12] and references quoted therein), the scattering properties of rectangular wells have been underestimated. Quite recently, however, nonevanescant

propagation has been predicted for potential wells [13]. In contradistinction with the tunneling exponential attenuation, the scattering at quantum wells attenuates the outgoing wave packets only because of the multiple reflections at the well boundaries. Negative phase times are then expected under certain conditions of the incident energy and the thickness of the well [13,14], a phenomenon which should be observable for electromagnetic wave propagation [15]. Thereby, rectangular wells may lead to much larger advancements than rectangular barriers in the context of traversal times [16].

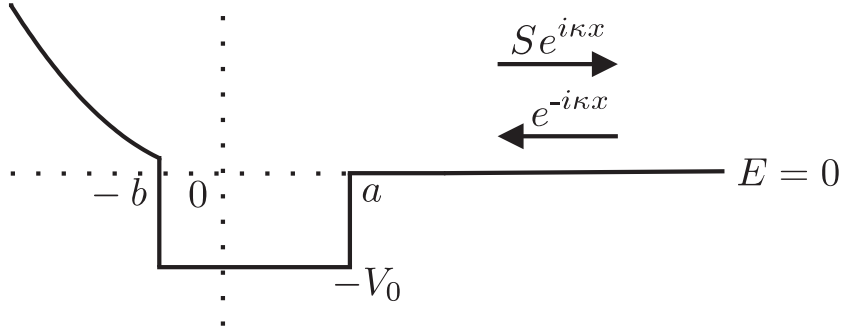


Figure 1: Schematic representation of the one-dimensional semi-harmonic square well as a function of the dimensionless position x . The wave e^{-ikx} colliding the well from the right is reflected to give $Se^{i\kappa x}$, with $S = e^{i\delta}$ the reflection amplitude and $\delta(E)$ the reflection phase shift.

The purpose of this contribution is to report negative time delay for a one-dimensional well which reduces the scattering process to the case of purely wave reflection. The absolute value of this negative time delay becomes larger as the energy of the incident particle becomes smaller. To begin with, consider the stationary Schrödinger equation $(H - E)\psi(x) = 0$, where $V(x)$ is the one-dimensional potential depicted in Figure 1. This last is a rectangular well in a semi-harmonic background integrated by zero potential energy (flat potential) at the right and a harmonic-like potential at the left of the well. Our model corresponds to a system (the rectangular well) embedded in an environment (the parabolic plus flat potentials), and the issue is the study of the modifications on the physical properties of the system due to the environment [17]. For instance, the number $N + 1$ of bound states $\psi_n(x)$, $n = 0, 1, \dots, N$, is determined by the area $A = (a + b)V_0$ of the rectangular well. Here, $a + b$ and $-V_0$ are respectively the width and depth of the well with $V_0 > 0$, $a \geq 0$, and $b \geq 0$. Once the semi-harmonic background is added, the number $N + 1$ is preserved but the corresponding energies E_0, E_1, \dots, E_N , are displaced towards the positive threshold. This last property does not depend on the geometry of the rectangle; the wells having the same area admit the same number of bound states. In this context, remark that the wells of unit area $V_0 = a + b$ admit only one bound state and constitute a family of compact support functions which converge to the delta well in the sense of distribution theory [18]. Then, the single bound state (dimensionless) energy $E_0 = -0.25$ of the delta well becomes less negative $E_0 = -0.0797104$ in the presence of the semi-harmonic background [17] (compare with [19]).

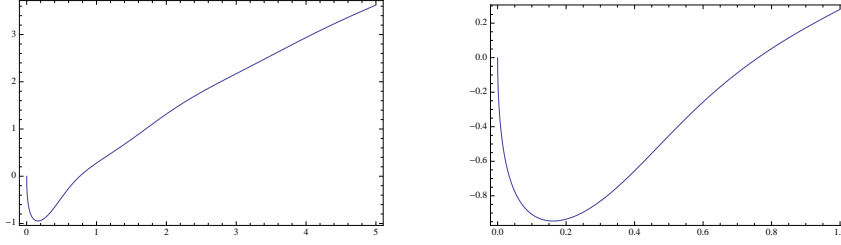


Figure 2: The reflection phase shift $\delta(E)$ of a semi-harmonic well of unit area as a function of the dimensionless energy E for the parameters $a = b = 5/2$, and $V_0 = 1/5$. A detail of the behavior of $\delta(E)$ for low energies is shown at the right.

On the other hand, the isolated resonances of a rectangular well are easily identified by expressing the transmission amplitude T as a superposition of Breit-Wigner distributions [20]. The center $E_k^r > 0$ and width Γ_k of each of these peaks define the resonance complex eigenvalue $\epsilon_k = E_k^r - i\Gamma_k/2$, and induce time delays in the scattering process [21]. A rapid increasing of the transmission phase is then expected in the vicinity of the resonance position E_k^r . According to Wigner, the increases of the phase should be balanced by the appropriate decreases [8]. Therefore, the slope of the transmission phase can be even negative in order to compensate for the phase increases associated with each of the resonances. This effect is more important near the energy threshold, below the position of the first Breit-Wigner peak of T [14]. In other words, the negative phase times predicted in [13] are in complete agreement with the conditions to get at least one isolated resonance in rectangular wells [14, 20]. If the semi-harmonic environment is activated, all the scattering states become more excited and their wave functions cancel at $x = -\infty$. As the potential includes neither sources nor shrinks, the probability is conserved and all the incoming waves are reflected. Then, the reflection phase shift $\delta(E)$ encodes all the information of the scattering process. This phase is depicted in Fig. 2 for a unit area semi-harmonic square well with $a = b = 5/2$. Notice the strong negative slope in the interval of dimensionless energies $(0, 0.16208517)$, so that negative time delay is expected for wave packets colliding the well from the right at the appropriate energy.

The straightforward calculation shows that the phase time is given by $\tau_W = \tau_p - \tau_E$, with $\tau_p = 2a/v_g$ the classical flight time to traverse a distance $2a$, and the time delay τ_E written in the form

$$\tau_E = \frac{1}{v_g} \frac{\partial}{\partial k} \left[\arctan \left(\frac{2\phi_1\phi_2}{\phi_1^2 - \phi_2^2} \right) \right].$$

Here the functions ϕ_1 and ϕ_2 are given by

$$\phi_1 = -\frac{q}{2} \sin 2qa + \frac{\psi'}{\psi} \Big|_{x=-a} \cos 2qa, \quad \phi_2 = -k \cos 2qa - \frac{1}{q} \frac{\psi'}{\psi} \Big|_{x=-a} \sin 2qa,$$

with

$$\psi(x) = e^{-x^2/2} \left[{}_1F_1 \left(\frac{1-k^2}{4}, \frac{1}{2}; x^2 \right) + 2x \frac{\Gamma(\frac{3-k^2}{4})}{\Gamma(\frac{1-k^2}{4})} {}_1F_1 \left(\frac{3-k^2}{4}, \frac{3}{2}; x^2 \right) \right],$$

a	E_a	a	E_a
2.5	0.03406092	1.0	0.10100123
2.0	0.05056413	0.5	0.16473112
1.5	0.07205970	0.0	0.45727096

Table 1: The (dimensionless) energy E_a defining the change of sign in the time delay for a semi-harmonic well of unit area.

and $q = \sqrt{V_0 + k^2}$. The expression ${}_1F_1(a, c; z)$ stands for the confluent hypergeometric function.

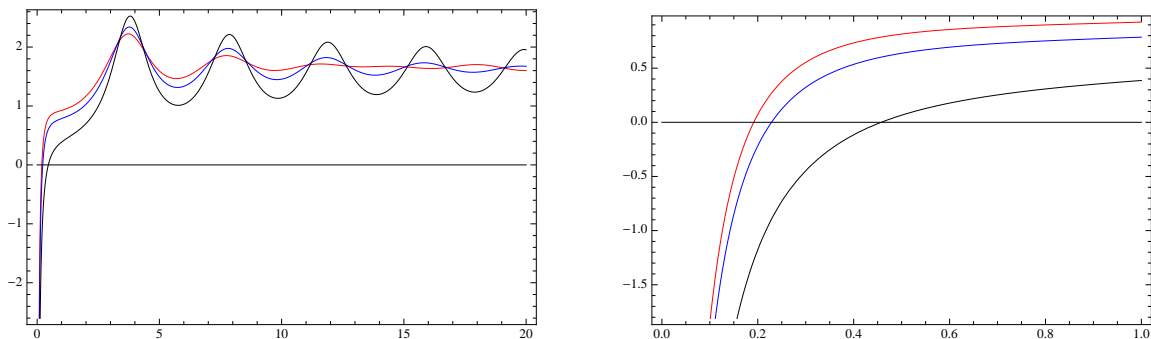


Figure 3: Time delay τ_E of a unit area semi-harmonic well for $a = 0.4$ (red curve), $a = 0.03$ (blue curve) and $a = 0$ (black curve). A detail of the behavior at low energies is shown at the right.

Figure 3 shows the behavior of the time delay τ_E for some semi-harmonic wells of unit area but different geometries. Given a , there is an interval of scattering energies $(0, E_a)$ where τ_E is negative (for definite values see Table 1). In this interval, the absolute value of the negative time delay becomes larger as the incident energy becomes smaller. Thus, it is clear the dependence of τ_E on the energy E of the incident particles and on the rectangular well thickness $2a$. For a given value of a , the maxima of the time delay are localized at the real part of the resonance eigenvalues $\epsilon_k = E_k^r - i\Gamma_k/2$, as expected. The energies E_k^r are displaced to more excited values as $a \rightarrow 0$. In the very limit $a = 0$, the time delay changes its sign at the scattering energy $E_{a=0} = 0.45727096$ and oscillates around the asymptotic value $\frac{\pi}{2}$ for $E > E_{a=0}$. It should be pointed out that the interval of scattering energies $(0, E_{a=0})$ is the largest one in which τ_E is negative for any of the unit area semi-harmonic wells (see Table 1 and Figure 3).

Let us close this contribution with some remarks on the optical analogs applied in the study of particles passing through a rectangular well [13, 15]. Of particular interest, negative phase times have been confirmed for electromagnetic wave propagation in waveguides filled with different dielectrics [15]. The negative time delay τ_E of the semi-harmonic wells could be studied in a similar way by taking $b = 0$ and $a \geq 0$. Once the energy baseline of the rectangular well is shifted by the constant value $E_0 = \hbar\omega_0$, the cutoff frequency ω_0 of the first waveguide section is defined. Then, waveguide sections with different cutoff

frequencies can be constructed to approximate the parabolic part of the potential by a series of Riemann rectangles. As a result, the semi-harmonic well can be connected to a piecewise frequency $\omega_c(x)$. Following [15], the solution to the propagation problem (i.e., the Helmholtz equation for ω_c) is obtained if the wave functions and the electromagnetic fields satisfy identical boundary conditions. Further details will be given elsewhere.

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